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UNIFIED COMMON FIXED POINT THEOREM IN 2- METRIC SPACES

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Abstract. The purpose of this paper is to prove some common fixed point theorems for four self maps in complete 2-metric spaces by employing the notion of weakly compatible mappings. Our results extend and generalize the results of Iseki (Fixed point theorems in 2-metric spaces, Math Seminar Notes, Kobe Uni. 3, 1975, 133 - 136) and several other authors.

Keywords: Fixed point, 2-metric space, weakly compatible, contractive modulus.

AMS Subject Classification: 54H25, 47H10.

1. Introduction

Fixed point theory has many applications, including variational and linear inequalities, optimization, approximation theory and minimum norm problem. Banach [1] proved the famous and well known Banach contraction principle concerning the fixed point of contraction mappings defined on a complete metric space. This theorem has been generalized and extended by many authors (see: [7, 8]).

In 1963, Gahler [5] introduced the generalization of metric space and called it 2-metric space. Let X be a set consisting at least three points. 2-metric on X is a function $\rho : X \times X \times X \to IR^{+}$ which satisfies the following conditions:

1. To each pair of points $a, b \in X$ with $a \neq b$, there exists a point $c \in X$ such that $\rho(a,b,c) \neq 0$;
2. $\rho(a,b,c) = 0$, when at least two of points are equal;
3. $\rho(a,b,c) = \rho(b,c,a) = \rho(c,a,b), \forall a,b,c \in X$
4. $\rho(a,b,c) \leq \rho(a,b,d) + \rho(a,d,c) + \rho(d,b,c), \forall a,b,c,d \in X$.

Here the 2 metric $\rho(x, y, z)$ represents the area of triangle spanned by x, y, z

Examples of 2-metric space are:

Example 1. [5] A circle in the Euclidean space R^2 is a 2-metric space.

Example 2. [5] Define d on $R^{+} \times R^{+} \times R^{+}$ as

$d(x,y,z)=\min\{|x-y|,|y-z|,|z-x|\}$.

Fixed Point Theory in 2-metric space has been proved initially by Iseki [9]. After that several authors ([12, 19, 22]) proved fixed point results in the setting of 2-metric space.

In 1979, Fisher [4] gave common fixed point using commuting mapping. Jungck [10] and Kubiak [15] also prove some results using commuting and semi-commuting mapping.

In 1992, Murthy [17] used compatible type mapping to prove fixed point results which is more general than commuting and semi-commuting maps.

After that in 1978, Khan [13] proved a result by taking a uniformly convergent sequence of 2-metrics in X .

In 1977, Fisher [3] proved the following result in metric space:

Theorem 1. [3] Let f be a self map on compete metric space (X,ρ) such that $\rho^2(fx,fy)\le\alpha\rho(x,fx)\rho(y,fy)-\beta\rho(x,fy)\rho(y,fx)$, $\forall\, x,y\in X$ and for some nonnegative constants α,β with $\alpha<1$. Then f has a fixed point in X . Moreover, if further $\beta<1$, then f has a unique fixed point in X .

Naidu and Prasad [18] in 1986 generalize the result of [3] in 2-metric space.

Further, in 1989, Bijendra [2] introduced the concept of semi-compatibility in 2-metric space and prove some fixed point results which improves the results of Kang et al. [12]. Also, Gupta et al. [21], [20] proved a result by using the concept of weak compatibility and property α. Gupta [6] in 2012, proved fixed point results using A-contraction in the setting of 2-metric space.

In 2011, Mehta et al. [16] proved fixed point result using weakly contractive condition and contractive modulus property in the setting of metric space. Also in 2014, Gupta et al. [11] showed result employing the same property in complete metric space.

In this paper, we prove a common fixed point result for four mappings by using weakly compatible property and contractive modulus.

2. Preliminaries

Definition 1. [9] A sequence $\{x_n\}$ said to be a Cauchy sequence in 2-metric space X , if for each $a\in X$ there exists $n_0\in X$, $\lim_{n,m\to\infty} d(x_n,x_m,a)=0$, $\forall n,m\ge n_0$.

Definition 2. [9] A sequence { x_n } in 2-metric space X is convergent to an element $x\in X$ if for each $a\in X$, $\lim_{n\to\infty} d(x_n,x,a)=0$.

Definition 3. [9] A complete 2-metric space is one in which every Cauchy sequence in X converges to an element of X .

Definition 4. [4] Let A and S be self mappings on a 2-metric space then, A and S are said to be weakly compatible if they commute at their coincidence point. i.e. If $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

Definition-5. [18] Two self maps f and g of a 2-metric space (X,d) are called compatible if $\lim_{n\to\infty} d\left(fgx_n, gfx_n, a\right) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition-6. [18] Two self maps f and g of a 2-metric space (X,d) are called non compatible if $\exists$ at least one sequence $\{x_n\}$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$. But $\lim_{n\to\infty} d\left(fgx_n, gfx_n, a\right)$ is either non zero or non – existent.

Definition-7. [4] Two self maps f and g are said to be commuting if $fgx = gfx \ \forall x \in X$.

Definition-8. [4]Let f and g be two self maps on a set X, if $fx = gx \ \forall x \in X$, then x is called coincidence point of f and g.

Definition-9. [16] A function $\phi : [0,\infty) \to [0,\infty)$ is said to be contractive modulus if $\phi(t) < t$, for $t > 0$.

3. Main result

Theorem 2. Let F, G, S and T be four self mappings on 2-metric space (X,d) satisfying the following conditions:

1. The pair (F,S) and (G,T) are weakly compatible,
2. $F(X) \subseteq T(X)$ and $G(X) \subseteq S(X)$ are closed subset of X,
3. $d(Fx,Gy,t) \le \phi[\min\{d(Sx,Ty,t), d(Fx,Sx,t), d(Gy,Ty,t), d(Fx,Ty,t), d(Sx,Gy,t)\}]$,

where ϕ is a contractive modulus.

Then the maps F, G, S and T have a unique common fixed point in X.

Proof. Let $\{y_n\}$ be a sequence in X such that $y_n = Fx_n = Tx_{n+1}$ and $y_{n+1} = Gx_{n+1} = Sx_{n+2}$, by (3)

$$d(y_n, y_{n+1}, t) = d(Fx_n, Gx_{n+1}, t)$$
$$\le \phi[\min\{d(Sx_n, Tx_{n+1}, t), d(Fx_n, Sx_n, t), d(Gx_{n+1}, Tx_{n+1}, t), d(Fx_n, Tx_{n+1}, t), d(Sx_n, Gx_{n+1}, t)$$
$$\le \phi[\min\{d(y_{n-1}, y_n, t), d(y_n, y_{n-1}, t), d(y_{n+1}, y_n, t), d(y_n, y_n, t), d(y_{n-1}, y_{n+1}, t)\}]$$
$$\le \phi[\min\{d(y_{n-1}, y_n, t), d(y_n, y_{n+1}, t)\}] \le \phi[d(y_n, y_{n+1}, t)]$$

Thus $d(y_n, y_{n+1}, t) \le \phi[d(y_n, y_{n+1}, t)]$.

But ϕ is a contractive module therefore $\phi[d(y_n, y_{n+1}, t)] < d(y_n, y_{n+1}, t)$ and this is possible only if $\lim_{n\to\infty} d(y_n, y_{n+1}, t) = 0$.

Now we show that $\{y_n\}$ is a Cauchy sequence in X. If not $\exists\ \varepsilon > 0$ such that $m < n < N, d(y_n, y_m, t) \geq \varepsilon,$ but $d(y_{n-1}, y_m, t) < \varepsilon$ and $\varepsilon \leq d(y_m, y_n, t) =$ $d(Fx_m, Gx_n, t)$

$\leq \phi[\min\{d(Sx_m, Tx_n, t), d(Fx_m, Sx_m, t), d(Gx_n, Tx_n, t), d(Fx_m, Tx_n, t), d(Sx_m, Gx_n, t)\}]$

$\leq \phi[\min\{d(Y_{m-1}, Y_{n-1}, t), d(Y_m, Y_{n-1}, t), d(y_n, y_{n-1}, t), d(y_m, y_{n-1}, t), d(y_{m-1}, y_n, t)\}]$

$\leq \phi[\min\{\varepsilon, \varepsilon, 0, \varepsilon, \varepsilon\}]$, This gives $\varepsilon \leq \phi(\varepsilon)$.

But ϕ is a contractive module therefore $\phi(\varepsilon) < \varepsilon,$ from this one can get $\varepsilon < \varepsilon,$ this is a contradiction, hence $\{y_n\}$ is a Cauchy sequence. Since X is complete therefore there exists a point z in X such that $\lim_{n\to\infty} y_n = z,$ this gives, $\lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Sx_n = z = \lim_{n\to\infty} Fx_n = \lim_{n\to\infty} T_n$. Since $F(X) \subseteq T(X), \exists$ a point $\alpha \in X$ $s.t.$ $z = T\alpha.$

If $z \neq G\alpha,$ using (3) we get $d(G\alpha, z, t) = d(G\alpha, Fx_n, t)$

$\leq \phi[\min\{d(Sx_n, T\alpha, t), d(Fx_n, Sx_n, t), d(G\alpha, T\alpha, t), d(Fx_n, T\alpha, t), d(Sx_n, G\alpha, t)$

$\leq \phi[\min\{d(z, z, t), d(z, z, t), d(G\alpha, z, t), d(z, z, t), d(z, G\alpha, t) \leq \phi[d(G\alpha, z, t)].$

This implies $d(G\alpha, z, t) \leq \phi[d(G\alpha, z, t)].$ But ϕ is a contractive modulus, this gives $\phi[d(G\alpha, z, t)] < d(G\alpha, z, t),$ this is a contradiction. Thus $G\alpha = z = T\alpha.$

Thus, α is a co-incidence point of G and T and (G,T) is weakly compatible, we get, $GT\alpha = TG\alpha \Rightarrow Gz = Tz.$ Now $G(X) \subseteq S(X)$ therefore there exists a point $w \in X$ $s.t.$ $Sw = z$ if $Fw \neq z.$

Using (3), $d(Fw, z, t) = d(G\alpha, Fw, t)$

$\leq \phi[\min\{d(Sw, T\alpha, t), d(Fw, Sw, t), d(G\alpha, T\alpha, t), d(Fw, T\alpha, t), d(Sw, G\alpha, t)$

$\leq \phi[\min\{d(z, z, t), d(Fw, z, t), d(z, z, t), d(Fw, z, t), d(z, z, t) \leq \phi[d(Fw, z, t)],$

this gives $d(Fw, z, t) \leq \phi[d(Fw, z, t)].$

But ϕ is a contractive modulus therefore $\phi[d(Fz, z, t)] < d(Fz, z, t)$ this is a contradiction.

So $Fw = z = Sw,$ hence w is a co-incidence point of F and S. Since (F, S) is weakly compatible therefore $FSw = SFw \Rightarrow Fz = Sz.$

Now if $Fz \neq z$ then by using (3) we can get,

$d(Fz, z, t) = d(Fz, G\alpha, t)$

$\leq \phi[\min\{d(Sz, T\alpha, t), d(Fz, Sz, t), d(G\alpha, T\alpha, t), d(Fz, T\alpha, t), d(Sz, G\alpha, t)\}]$

$\leq \phi[\min\{d(Sz, z, t), d(Fz, Sz, t), d(z, z, t), d(z, z, t), d(Sz, z, t)\}].$

Since $Fz = Sz,$ therefore $d(Fz, z, t) \le \phi[d(Fz, z, t)]$. Also ϕ is a contractive modulus. Thus $\phi[d(Fz, z, t)] < d(Fz, z, t).$ This is a contradiction. Hence $Fz = Sz = z.$ Now if $Gz \ne z$ then by using (3), we get $d(z, Gz, t) = d(Fz, Gz, t)$

$\le \phi[\min\{d(Sz, Tz, t), d(Fz, Sz, t), d(Gz, Tz, t), d(Fz, Tz, t), d(Sz, Gz, t)\}]$

$\le \phi[\min\{d(z, Tz, t), d(z, z, t), d(Gz, Tz, t), d(z, Tz, t), d(z, Gz, t)\}].$

And $Gz = Tz \Rightarrow d(z, Gz, t) \le \phi[d(z, Gz, t)]$ and ϕ is a contractive modulus, therefore $\phi[d(z, Gz, t)] < d(z, Gz, t),$ which is a contradiction. So $Gz = z = Tz,$ hence we have $Gz = Tz = Fz = Sz = z.$

Hence F, S, T, G have a common fixed point in X .

Now we prove uniqueness.

Let there be another point say w s.t. $w \ne z$, then by (3)

$d(Fz, Gw, t) \le \phi[\min\{d(Sz, Tw, t), d(Fz, Sz, t), d(Gw, Tw, t), d(Fz, Tw, t), d(Sz, Gw, t)\}]$

$d(z, w, t) \le \phi[\min\{d(z, w, t), d(z, z, t), d(w, w, t), d(z, w, t), d(z, w, t)\}]$

$\Rightarrow d(z, w, t) \le \phi[d(z, w, t)]$

Since, ϕ is a contractive modulus, we get $\Rightarrow \phi[d(z, w, t)] < d(z, w, t)$, which is a contradiction.

Therefore fixed points are unique. This proves the Theorem 2.1.

Corollary 1. Let F, G, S and T be four self mappings of a 2-metric space (X, d) satisfying the following conditions:

1. The pairs (F, S) and (G, T) are weakly compatible.
2. $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Gy_n = \lim_{n\to\infty} Ty_n = z$ for some z in X
3. $d(Fx, Gy, t) \le \phi[\min\{d(Sx, Ty, t), d(Fx, Sx, t), d(Gy, Ty, t), d(Fx, Ty, t), d(Sx, Gy, t)\}],$

where ϕ is a contractive modulus. Then the maps F, G, S and T have a unique common fixed point in X .

Proof. Using condition (2), since $\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Gy_n = \lim_{n\to\infty} Ty_n = z$ for some z in X since $\lim_{n\to\infty} Ty_n = z$ then there exists a point $\alpha \in X$ $s.t.$ $z = T\alpha,$ refers this to the proof of theorem 3.1, we have corollary 1.

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2-ölçülü metrik fəzalarda tərpənməz nöqtə haqqında vahid ümumi teorem

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XÜLASƏ

Bu işin məqsədi 2 ölçülü metrik fəzalarda zəif uyuşan inikas anlayışından istifadə etməklə, 4 ayrı misal üçün tərpənməz nöqtə haqqında bəzi teoremlərin isbat edilməsidir. Bu nəzticələr Iseki və digər bəzi müəlliflərin nəticələrini genişləndirir və ümumiləşdirir.

Açar sözlər: tərpənməz nöqtə, 2 ölçülü metrik fəzalar, zəif uyğunluq, sıxılan modullar

Единая общая теорема о неподвижной точке в 2-мерном метрическом пространстве

Нэвин Гулати, Вишал Гупта, Равиндер Кумар, Сима Деви

РЕЗЮМЕ

Целью данной работы является доказать некоторые общие теоремы о неподвижной точке для четырех самостоятельных примеров в 2-мерном метрическом пространстве с использованием понятие слабо совместимых отображений. Наши результаты расширяют и обобщают результаты Iseki и ряда других авторов.

Ключевые слова: неподвижная точка, 2-мерное метрическое пространство, слабо совместимость, сжимающие модули.